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Orthogonal Polynomials of Sobolev Type on the Unit Circle

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In this paper we study algebraic and asymptotic properties of orthogonal polynomials with respect to an inner product in the linear space $\mathbb{P}$ of complex polynomials defined by

$$\langle P(z), Q(z)\rangle = \varphi_u(P(z), Q(z)) + \lambda^{-1}P'(a)\,\overline{Q'(a)},$$

where φ_u is a regular and Hermitian form, $\lambda \in \mathbb{R}\setminus\{0\}$, and $|a| = 1$. Major emphasis is given to the positive definite case.

1. INTRODUCTION

The study of inner products of Sobolev type with respect to a finite and positive Borel measure supported on a bounded or unbounded interval and the corresponding sequences of orthogonal polynomials has been developed during recent years.

After the pioneering paper by Koekoek [10], the contributions by several members of the Delft group (Bavinck and Meijer, among others) deal with some classical weight functions (essentially those corresponding to Laguerre and Gegenbauer polynomials) which are modified by addition the derivatives at the endpoints of the interval of orthogonality.

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Marcellán and Ronveaux [13] analyze the question in a more general situation. They consider inner products of the form

$$\langle f, g\rangle = \int_I f(x)\, g(x)\, w(x)\, dx + Mf^{(r)}(c)\, g^{(r)}(c), \tag{I}$$

where w is a weight function on the interval I, $M \in \mathbb{R}^+$, $c \in \mathbb{R}$, and $r \in \mathbb{N}$. The basic tool is the symmetry of the multiplication operator H defined in the linear space $\mathbb{P}$ of complex polynomials by $(Hp)(x) = (x-c)^{r+1}\, p(x)$. As a simple consequence, a recurrence relation can be deduced for the sequence of orthogonal polynomials with respect to (I).

An alternative and fruitful method involving matrix techniques is developed by Meijer [15] in a recent paper for $r=1$. The author presents a natural extension of the Laguerre case and the location of zeros of orthogonal polynomials in terms of the position of point c with respect to the support of the measure is given. More recently, a more detailed analysis appears in [2] for inner products of the form

$$\langle f, g\rangle = \int_I f(x)\, g(x)\, d\mu(x) + Mf(c)\, g(c) + Nf'(c)\, g'(c). \tag{II}$$

Moreover, an analysis of relative asymptotic properties for sequences of orthogonal polynomials of Sobolev type when the measure μ belongs to the Nevai class has been presented in [14].

In this paper we deal with the consideration of inner products of Sobolev type on the unit circle T. Given a finite and positive Borel measure μ on T, we consider the following inner product on the linear space $\mathbb{P}$

$$\langle P(z), Q(z)\rangle = \int_T P(z)\, \overline{Q(z)}\, d\mu + \lambda^{-1} P^{(k)}(a)\, \overline{Q^{(k)}(a)},$$

where $\lambda \in \mathbb{R}\backslash\{0\}$ and $a \in T$. This situation corresponds to a Sobolev inner product

$$\int_T P(z)\, \overline{Q(z)}\, d\mu + \int_T P^{(k)}(z)\, \overline{Q^{(k)}(z)}\, d\mu_k$$

with μ_k a singular measure on the unit circle whose support is $\{a\}$. In [3, 4], we have analyzed the case $k=0$, taking into account the fact that the moment matrix is Toeplitz and positive definite.

The structure of the paper is as follows. In Section 2 we consider a regular linear functional u which is Hermitian on the linear space of Laurent polynomials Λ. Next we define a Hermitian form $\langle\ ,\ \rangle$ such that

$$\langle P(z), Q(z)\rangle = u\left(P(z)\, \bar{Q}\left(\frac{1}{z}\right)\right) + \lambda^{-1} P'(a)\, \overline{Q'(a)}, \tag{III}$$

where $\lambda \in \mathbb{R}\setminus\{0\}$ and $a \in T$. In Theorem 2.2, a necessary and sufficient condition for the existence of a sequence of monic orthogonal polynomials with respect to (III) is given. In these conditions, an algebraic relation between the corresponding sequences of monic orthogonal polynomials for u and $\langle\ ,\ \rangle$ is obtained (Theorem 2.3). As a consequence, a connection with some polynomial modifications of the linear functional u is presented (Theorem 2.5). Then, the concept of para-orthogonality [9] plays a very important role (Theorem 2.7). In Section 3 we analyze the situation when u is a positive definite linear functional. Then, we can consider orthonormal polynomials and some useful results appear. In Section 4, we obtain some relative asymptotic properties in the direction pointed out in [14] for bounded intervals and measures belonging to the Nevai class. Our main result (Theorem 4.3) holds for $\mu' > 0$.

2. Algebraic Properties

Let $\Lambda = \operatorname{span}\{z^k\}_{k \in \mathbb{Z}}$ be the space of Laurent polynomials. We denote by $\mathbb{P} = \operatorname{span}\{z^k\}_{k \in \mathbb{N}}$ the space of polynomials with complex coefficients. Let $u: \Lambda \to \mathbb{C}$ be a Hermitian linear functional, that is, $u(z^n) = \overline{u(z^{-n})}$ for all $n \geqslant 0$.

Definition 2.1. A linear functional defined as above is called regular (resp. positive definite) if the principal submatrices $(u_{j-i})_{i=0,..,n,\, j=0,..,n} = (u(z^{j-i}))_{i=0,..,n,\, j=0,..,n}$ of the moment matrix are nonsingular (resp. positive definite).

Consider the bilinear form φ_u defined by $\varphi_u(P(z), Q(z)) = u(P(z)\,\bar{Q}(1/z))$ where $P(z), Q(z) \in \mathbb{P}$. For $\lambda \in \mathbb{R}\setminus\{0\}$ and $a \in \mathbb{C}$ with $|a| = 1$, we define the following Hermitian bilinear form:

$$\langle P(z), Q(z)\rangle = \varphi_u(P(z), Q(z)) + \lambda^{-1} P'(a)\,\overline{Q'(a)}.$$

The Gram matrix associated to $\langle\ ,\ \rangle$ with respect to the canonical basis $\{z^n\}_{n \in \mathbb{N}}$ is not Toeplitz. Also we say that a bilinear Hermitian form is regular if the Gram matrix associated to the form has nonsingular principal submatrices. Taking into account previous definitions, we have

Theorem 2.2. *If u is a regular functional then*

$$\langle\ ,\ \rangle \text{ is regular} \Leftrightarrow \lambda + K_{n-1}^{(1,1)}(a, a) \neq 0 \qquad (n \geqslant 1),$$

where $K_{n-1}^{(1,1)}(a, a) = \sum_{j=1}^{n-1} |\phi_j'(a)|^2/e_j$, $\{\phi_n(z)\}_{n \in \mathbb{N}}$ *is the monic orthogonal* *polynomial sequence* (MOPS) *related to u and* $e_j = u(\phi_j(z)\,\overline{\phi_j}(1/z))$. *Furthermore, if* $\{\psi_n(z)\}_{n\in\mathbb{N}}$ *is the MOPS related to* $\langle\ ,\ \rangle$ *then*

$$\psi_n(z) = \phi_n(z) - \frac{\phi_n'(a)}{\lambda + K_{n-1}^{(1,1)}(a,a)} K_{n-1}^{(0,1)}(z,a), \qquad (1),$$

where $K_{n-1}^{(0,1)}(z,a) = \sum_{j=1}^{n-1} \phi_j(z)\,\overline{\phi_j'(a)}/e_j$.

Proof. $(\Rightarrow)$ Assume $\langle\ ,\ \rangle$ is regular and let $\{\psi_n(z)\}_{n\in\mathbb{N}}$ be the corresponding MOPS, that is, $\langle \psi_n(z), \psi_m(z)\rangle = a_n\,\delta_{nm}$ with $a_n \neq 0$. Under these conditions $\psi_n(z) = \phi_n(z) + \sum_{j=0}^{n-1} b_{nj}\phi_j(z)$ where

$$b_{nj} = \frac{\varphi_u(\psi_n(z), \phi_j(z))}{\varphi_u(\phi_j(z), \phi_j(z))} = \frac{-\lambda^{-1}\psi_n'(a)\,\overline{\phi_j'(a)}}{e_j} \qquad (0 \leqslant j \leqslant n-1).$$

Then $\psi_n(z) = \phi_n(z) - \lambda^{-1}\psi_n'(a)\,K_{n-1}^{(0,1)}(z,a)$.

On the other hand, if we take derivatives in this expression and we evaluate at $z=a$ we get $\psi_n'(a)[\lambda + K_{n-1}^{(1,1)}(a,a)] = \lambda\phi_n'(a)$. Now $\langle \psi_n(z), \phi_n(z)\rangle = \varphi_u(\phi_n(z), \phi_n(z)) + \lambda^{-1}\psi_n'(a)\,\overline{\phi_n'(a)}$, that is,

$$e_n^{-1}\langle \psi_n(z), \phi_n(z)\rangle = 1 + e_n^{-1}\,\lambda^{-1}\psi_n'(a)\,\overline{\phi_n'(a)},$$

and since $\lambda = \lambda + K_0^{(1,1)}(a,a) \neq 0$, then

$$\begin{aligned} e_1^{-1}\langle \psi_1(z), \phi_1(z)\rangle &= 1 + e_1^{-1}\lambda^{-1}\psi_1'(a)\,\overline{\phi_1'(a)} = 1 + e_1^{-1}\frac{|\phi_1'(a)|^2}{\lambda + K_0^{(1,1)}(a,a)} \\ &= \frac{\lambda + K_1^{(1,1)}(a,a)}{\lambda + K_0^{(1,1)}(a,a)}. \end{aligned}$$

From the regularity of $\langle\ ,\ \rangle$ it follows that

$$\langle \psi_1(z), \phi_1(z)\rangle = \langle \psi_1(z), \psi_1(z)\rangle \neq 0,$$

and therefore $\lambda + K_1^{(1,1)}(a,a) \neq 0$. In a similar way, by induction we obtain the result.

$(\Leftarrow)$ If $\lambda + K_{n-1}^{(1,1)}(a,a) \neq 0$ for all $n \geqslant 1$, we define a sequence of monic polynomials $\{\psi_n(z)\}_{n\in\mathbb{N}}$ as

$$\psi_n(z) = \phi_n(z) - \frac{\phi_n'(a)}{\lambda + K_{n-1}^{(1,1)}(a,a)} K_{n-1}^{(0,1)}(z,a).$$

It holds that $\langle \psi_n(z), \phi_m(z)\rangle = 0$ for $m < n$ and

$$\langle \psi_n(z), \phi_n(z)\rangle \neq 0 \qquad \forall n \in \mathbb{N}.$$

Indeed, for $m<n$

$$\begin{aligned}\langle\psi_n(z),\phi_m(z)\rangle &= \varphi_u(\psi_n(z),\phi_m(z))+\lambda^{-1}\psi'_n(a)\,\overline{\phi'_m(a)}\\ &= -\frac{\phi'_n(a)\,\overline{\phi'_m(a)}}{\lambda+K^{(1,1)}_{n-1}(a,a)}-\frac{\lambda^{-1}\phi'_n(a)\,\overline{\phi'_m(a)}\,\lambda}{\lambda+K^{(1,1)}_{n-1}(a,a)}=0\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle\psi_n(z),\phi_n(z)\rangle &= \varphi_u(\psi_n(z),\phi_n(z))+\lambda^{-1}\psi'_n(a)\,\overline{\phi'_n(a)}\\ &= \varphi_u(\phi_n(z),\phi_n(z))+\frac{|\phi'_n(a)|^2}{\lambda+K^{(1,1)}_{n-1}(a,a)}\\ &= e_n\,\frac{\lambda+K^{(1,1)}_{n}(a,a)}{\lambda+K^{(1,1)}_{n-1}(a,a)}\neq 0\end{aligned}$$

from the hypothesis. ∎

THEOREM 2.3. *If $\{\phi_n(z)\}_{n\in\mathbb{N}}$, $\{\psi_n(z)\}_{n\in\mathbb{N}}$ are, respectively, the MOPS related to u and $\langle\ ,\ \rangle$, then*

$$(z-a)^2\,\psi_n(z)=q_{1,n}(z)\,\phi^*_n(z)+q_{2,n}(z)\,\phi_n(z), \tag{2}$$

where

$$\begin{aligned}q_{1,n}(z) &= (\bar a)^{n-3}\,e_n^{-1}\,\frac{\phi'_n(a)}{\lambda+K^{(1,1)}_{n-1}(a,a)}\\ &\quad\times([(n-1)\,\bar a\phi_n(a)-\phi'_n(a)]\,(z-a)-\phi_n(a))\\ q_{2,n}(z) &= (z-a)^2+e_n^{-1}\,\frac{a\phi'_n(a)}{\lambda+K^{(1,1)}_{n-1}(a,a)}\\ &\quad\times([a\,\overline{\phi_n(a)}-\overline{\phi'_n(a)}](z-a)+a^2\,\overline{\phi_n(a)}).\end{aligned}$$

Proof. From the Christoffel–Darboux formula [1; 5, p. 196, (1.21); 8, p. 10, (8.1); 12],

$$K_{n-1}(z,y)=e_n^{-1}\,\frac{[\phi^*_n(z)\,\overline{\phi^*_n(y)}-\phi_n(z)\,\overline{\phi_n(y)}]}{(1-z\bar y)}, \tag{3}$$

if we set $y=a$ and since $|a|=1$, we get

$$K_{n-1}(z,a)=ae_n^{-1}\,\frac{[\phi_n(z)\,\overline{\phi_n(a)}-\phi^*_n(z)\,\overline{\phi^*_n(a)}]}{z-a}.$$

(Recall that the $\overset{*}{n}$ operator is defined by $P^*(z)=z^n\bar{P}(1/z)$ for $P\in\mathbb{P}_n$ and the kernel $K_{n-1}(z, y)$ is defined by $K_{n-1}(z, y)=\sum_{j=0}^{n-1}\phi_j(z)\overline{\phi_j(y)}/e_j$). In the same way, by taking derivatives with respect to z in (3) and evaluating at $z=a$ we obtain

$$K_{n-1}^{(1,0)}(a, y)=e_n^{-1}\left(\bar{y}\,\frac{[\phi_n^*(a)\,\overline{\phi_n^*(y)}-\phi_n(a)\,\overline{\phi_n(y)}]}{(1-a\bar{y})^2}\right.$$
$$\left.+\frac{[\phi_n^{*\prime}(a)\,\overline{\phi_n^*(y)}-\phi_n'(a)\,\overline{\phi_n(y)}]}{(1-a\bar{y})}\right).$$

Taking conjugates,

$$e_n(z-a)^2\,K_{n-1}^{(0,1)}(z, a)=r_{1,n}(z)\,\phi_n(z)+t_{1,n}(z)\,\phi_n^*(z)$$

holds where

$$r_{1,n}(z)=(a\,\overline{\phi_n'(a)}-a^2\,\overline{\phi_n(a)})\,z-a^2\,\overline{\phi_n'(a)}$$
$$t_{1,n}(z)=(a^2\,\overline{\phi_n^*(a)}-a\,\overline{\phi_n^{*\prime}(a)})\,z+a^2\,\overline{\phi_n^{*\prime}(a)}.$$

Therefore, by substituting these expressions in (1) and taking into account that $\overline{\phi_n^*(a)}=(\bar{a})^n\,\phi_n(a)$ and $\phi_n^{*\prime}(a)=n\bar{a}(\phi_n^*(a)-a^{n-1}\overline{\phi_n'(a)}/n)$ [17], (2) follows. ∎

If we apply to (2) the $\overset{*}{n+2}$ operator we obtain

$$(z-a)^2\,\psi_n^*(z)=a^2q_{1,n}^*(z)\,z\phi_n(z)+a^2q_{2,n}^*(z)\,\phi_n^*(z),$$

and so, using Cramer's rule,

$$\phi_n(z)=\frac{\begin{vmatrix}(z-a)^2\,\psi_n(z) & q_{1,n}(z)\\ (z-a)^2\,\psi_n^*(z) & a^2q_{2,n}^*(z)\end{vmatrix}}{\begin{vmatrix}q_{2,n}(z) & q_{1,n}(z)\\ a^2zq_{1,n}^*(z) & a^2q_{2,n}^*(z)\end{vmatrix}}.$$

Therefore, the denominator can be written as

$$R_{2,n}(z)(z-a)^2=a^2[q_{2,n}(z)\,q_{2,n}^*(z)-q_{1,n}(z)\,zq_{1,n}^*(z)]$$

Consequently, we have

$$R_{2,n}(z)\,\phi_n(z)=a^2q_{2,n}^*(z)\,\psi_n(z)-q_{1,n}(z)\,\psi_n^*(z),$$

and in the same way

$$R_{2,n}(z)\,\phi_n^*(z)=q_{2,n}(z)\,\psi_n^*(z)-a^2zq_{1,n}^*(z)\,\psi_n(z).$$

If we assume that the linear functionals $u_1=(z-a)(1/z-\bar{a})\,u$ and $u_2=(z-a)^2\,(1/z-\bar{a})^2\,u$ are regular [6], it is possible to give an alternative representation for expression (2). If we denote by $\{\phi_n^{1)}(z)\}_{n\in\mathbb{N}}$ and $\{\phi_n^{2)}(z)\}_{n\in\mathbb{N}}$ the MOPS related to u_1 and u_2 respectively, then the following is well-known:

THEOREM 2.4.

(i) $$(z-a)\,\phi_{n-1}^{1)}(z)=\phi_n(z)-\frac{\phi_n(a)}{K_{n-1}(a,a)}\,K_{n-1}(z,a);$$

(ii) $$\phi_{n-1}^{1)}(a)=\phi_n'(a)-\frac{\phi_n(a)}{K_{n-1}(a,a)}\,K_{n-1}^{(0,1)}(a,a);$$

(iii) $$K_{n-2}^{1)}(z,a)=\frac{1}{K_{n-1}(a,a)}\,\frac{\begin{vmatrix} K_{n-1}^{(0,1)}(z,a) & K_{n-1}^{(0,1)}(a,a) \\ K_{n-1}(z,a) & K_{n-1}(a,a)\end{vmatrix}}{(z-a)};$$

(iv) $$K_{n-2}^{1)}(a,a)=\frac{1}{K_{n-1}(a,a)}\begin{vmatrix} K_{n-1}^{(1,1)}(a,a) & K_{n-1}^{(0,1)}(a,a) \\ K_{n-1}^{(1,0)}(a,a) & K_{n-1}(a,a)\end{vmatrix};$$

where

$$K_{n-1}^{1)}(z,y)=\sum_{j=0}^{n-1}\frac{\phi_j^{1)}(z)\,\overline{\phi_j^{1)}(y)}}{e_j^{1)}} \qquad \text{and} \qquad e_j^{1)}=u_1(\phi_j^{1)}(z)\,\overline{\phi_j^{1)}}(1/z)).$$

Proof. See [6, and 7]. ∎

By using the identities of the previous theorem and taking into account that in the regular case $\phi_n(a)\neq 0\ \forall n$, with $|a|=1$ [11],

$$\begin{aligned}
(z-a)^2\,\phi_{n-2}^{2)}(z) &= (z-a)\,\phi_{n-1}^{1)}(z)-\frac{\phi_{n-1}^{1)}(a)}{K_{n-2}^{1)}(a,a)}\,(z-a)\,K_{n-2}^{1)}(z,a)\\
&= (z-a)\,\phi_{n-1}^{1)}(z)-\frac{\phi_{n-1}^{1)}(a)}{K_{n-2}^{1)}(a,a)}\,K_{n-1}^{(0,1)}(z,a)\\
&\quad+\frac{\phi_{n-1}^{1)}(a)\,K_{n-1}^{(0,1)}(a,a)\,K_{n-1}(z,a)\,\phi_n(a)}{K_{n-2}^{1)}(a,a)\,\phi_n(a)\,K_{n-1}(a,a)}\\
&= (z-a)\,\phi_{n-1}^{1)}(z)-\frac{\phi_{n-1}^{1)}(a)}{K_{n-2}^{1)}(a,a)}\,K_{n-1}^{(0,1)}(z,a)\\
&\quad+\frac{\phi_{n-1}^{1)}(a)\,K_{n-1}^{(0,1)}(a,a)}{K_{n-2}^{1)}(a,a)\,\phi_n(a)}\times[\phi_n(z)-(z-a)\,\phi_{n-1}^{1)}(z)].
\end{aligned}$$

From this

$$K_{n-1}^{(0,1)}(z,a)=\frac{K_{n-1}^{(0,1)}(a,a)}{\phi(a)}\phi_n(z)+\frac{K_{n-2}^{1)}(a,a)}{\phi_{n-1}^{1)}(a)}$$
$$\times\left(1-\frac{\phi_{n-1}^{1)}(a)\,K_{n-1}^{(0,1)}(a,a)}{K_{n-2}^{1)}(a,a)\,\phi_n(a)}\right)(z-a)\,\phi_{n-1}^{1)}(z)$$
$$-\frac{K_{n-2}^{1)}(a,a)}{\phi_{n-1}^{1)}(a)}(z-a)^2\phi_{n-2}^{2)}(z),$$

or equivalently

$$K_{n-1}^{(0,1)}(z,a)=\frac{K_{n-1}^{(0,1)}(a,a)}{\phi_n(a)}\left(\phi_n(z)+\left(\frac{K_{n-2}^{1)}(a,a)\,\phi_n(a)}{\phi_{n-1}^{1)}(a)\,K_{n-1}^{(0,1)}(a,a)}-1\right)\right.$$
$$\times(z-a)\,\phi_{n-1}^{1)}(z)-\frac{K_{n-2}^{1)}(a,a)\,\phi_n(a)}{\phi_{n-1}^{1)}(a)\,K_{n-1}^{(0,1)}(a,a)}$$
$$\left.\times(z-a)^2\,\phi_{n-2}^{2)}(z)\right). \tag{4}$$

Therefore from (1) follows

THEOREM 2.5.

$$\psi_n(z)=\mathscr{A}_n\phi_n(z)+\mathscr{B}_n(z-a)\,\phi_{n-1}^{1)}(z)+\mathscr{C}_n(z-a)^2\,\phi_{n-2}^{2)}(z), \tag{5}$$

where

$$\mathscr{A}_n=1-\frac{\phi_n'(a)\,K_{n-1}^{(0,1)}(a,a)}{\phi_n(a)\,(\lambda+K_{n-1}^{(1,1)}(a,a))}=1-\alpha_n$$

$$\mathscr{B}_n=\frac{\phi_n'(a)\,K_{n-1}^{(0,1)}(a,a)}{\phi_n(a)\,(\lambda+K_{n-1}^{(1,1)}(a,a))}-\frac{\phi_n'(a)\,K_{n-2}^{1)}(a,a)}{\phi_{n-1}^{1)}(a)(\lambda+K_{n-1}^{(1,1)}(a,a))}=\alpha_n-\beta_n$$

$$\mathscr{C}_n=\frac{\phi_n'(a)\,K_{n-2}^{1)}(a,a)}{\phi_{n-1}^{1)}(a)(\lambda+K_{n-1}^{(1,1)}(a,a))}=\beta_n.$$

Proof. By Substituting the expression (4) into (1), we obtain that $\psi(z)$ is a linear convex combination of $\phi_n(z)$, $(z-a)\,\phi_{n-1}^{1)}(z)$ and $(z-a)^2\,\phi_{n-2}^{2)}(z)$. ∎

Equation (5) depends on three families of orthogonal polynomials which are closely connected by standard polynomial transformations. This result is the analog of a result which has been obtained in [2] for the real case. In that situation the position of the mass point with respect to the support of the measure played a central role.

LEMMA 2.6.

$$\varphi_{u_1}(\psi_n(z), zP(z)) = 0 \qquad \forall P \in \mathbb{P}_{n-3} = \operatorname{span}\{z^k/0 \leqslant k \leqslant n-3\}.$$

Proof.

$$\begin{aligned}
\varphi_{u_1}(\psi_n(z), zP(z)) &= u_1\left(\psi_n(z)\frac{1}{z}\bar{P}\left(\frac{1}{z}\right)\right) = u\left((z-a)(1/z-\bar{a})\,\psi_n(z)\frac{1}{z}\bar{P}\left(\frac{1}{z}\right)\right) \\
&= u\left((1-a/z)(1/z-\bar{a})\,\psi_n(z)\,\bar{P}\left(\frac{1}{z}\right)\right) \\
&= au\left(\psi_n(z)(\bar{a}-1/z)(1/z-\bar{a})\,\bar{P}\left(\frac{1}{z}\right)\right) \\
&= -au\left(\psi_n(z)(1/z-\bar{a})^2\,\bar{P}\left(\frac{1}{z}\right)\right) \\
&= -a\varphi_u(\psi_n(z), (z-a)^2 P(z)) \\
&= -a\langle \psi_n(z), (z-a)^2 P(z)\rangle = 0 \qquad \forall P \in \mathbb{P}_{n-3}. \quad \blacksquare
\end{aligned}$$

THEOREM 2.7. *With the above notations,*

$$\psi_n(z) = \phi_n^{1)}(z) + M_n \phi_{n-1}^{1)}(z) + N_n \phi_{n-2}^{1)*}(z) \tag{6}$$

holds where

$$\begin{aligned}
M_n &= \frac{\phi_{n+1}(a)\,\overline{\phi_n(a)}}{e_n K_n(a,a)} - \frac{\phi_n'(a)\,\overline{\phi_{n-1}'(a)}}{(\lambda + K_{n-1}^{(1,1)}(a,a))\,e_{n-1}} \\
&\quad - a - \phi_{n+1}(0)\,\overline{\phi_n(0)}
\end{aligned}$$

$$\begin{aligned}
e_{n-1}\,\overline{\phi_{n-2}(0)}\,N_n &= \overline{\phi_{n-1}(a)}\left(\frac{\phi_{n+1}(a)}{K_n(a,a)} + \frac{M_n\phi_n(a)}{K_{n-1}(a,a)}\right) \\
&\quad - \phi_{n+1}(0)\,\overline{\phi_{n-1}(0)}\,e_n + \frac{\phi_n'(a)}{\lambda + K_{n-1}^{(1,1)}(a,a)} \\
&\quad \times \left(a\overline{\phi_{n-1}'(a)} - \overline{\phi_{n-1}'(a)}\,\phi_n(0)\,\overline{\phi_{n-1}(0)} - \frac{e_{n-1}}{e_{n-2}}\,\overline{\phi_{n-2}'(a)}\right).
\end{aligned}$$

Proof. From

$$\begin{aligned}
\mathbb{P}_n &= L[\phi_n^{1)}(z)] \oplus \mathbb{P}_{n-1} = L[\phi_n^{1)}(z), \phi_{n-1}^{1)}(z)] \oplus \mathbb{P}_{n-2} \\
&= L[\phi_n^{1)}(z), \phi_{n-1}^{1)}(z)] \oplus L[\phi_{n-2}^{1)*}(z)] \oplus z\mathbb{P}_{n-3},
\end{aligned}$$

by using the previous lemma, (6) follows. The coefficients may be computed as follows: Multiplying (6) by $z-a$, from Theorem 2.4(i) we get

$$\begin{aligned}(z-a)\,\psi_n(z) &= \phi_{n+1}(z) - \frac{\phi_{n+1}(a)}{K_n(a,a)} K_n(z,a)\\ &\quad + M_n\left(\phi_n(z) - \frac{\phi_n(a)}{K_{n-1}(a,a)} K_{n-1}(z,a)\right)\\ &\quad + N_n(z-a)\,\phi^*_{n-2}(z). \qquad (7)\end{aligned}$$

Then, if we apply the functional u

$$u\left((z-a)\,\psi_n(z)\,\overline{\phi_n}\left(\frac{1}{z}\right)\right) = \frac{-\phi_{n+1}(a)\,\overline{\phi_n(a)}}{K_n(a,a)} + M_n e_n \qquad (8)$$

holds. In accordance with (1),

$$\begin{aligned}(z-a)\,\psi_n(z) &= \phi_{n+1}(z) - \phi_{n+1}(0)\,\phi^*_n(z) - a\phi_n(z)\\ &\quad - \frac{\phi'_n(a)}{\lambda + K^{(1,1)}_{n-1}(a,a)}\left(\frac{\overline{\phi'_{n-1}(a)}}{e_{n-1}} z^n + \cdots\right).\end{aligned}$$

Therefore, substituting in (8) we get

$$-\phi_{n+1}(0)\,\overline{\phi_n(0)}\,e_n - a e_n - \frac{\phi'_n(a)\,\overline{\phi'_{n-1}(a)}\,e_n}{(\lambda + K^{(1,1)}_{n-1}(a,a))\,e_{n-1}} = -\frac{\phi_{n+1}(a)\,\overline{\phi_n(a)}}{K_n(a,a)} + M_n e_n$$

and M_n can be deduced inmediately.

In the same way, if we multiply by $\overline{\phi_{n-1}}\,(1/z)$ in (7) and apply u, we have

$$\begin{aligned}u\left((z-a)\,\psi_n(z)\,\overline{\phi_{n-1}}\left(\frac{1}{z}\right)\right) &= -\frac{\phi_{n+1}(a)\,\overline{\phi_{n-1}(a)}}{K_n(a,a)} - \frac{M_n\phi_n(a)\,\overline{\phi_{n-1}(a)}}{K_{n-1}(a,a)}\\ &\quad + N_n e_{n-1}\,\overline{\phi_{n-2}(0)}.\end{aligned}$$

On the other hand, if we compute the left-hand side of the above expression using (1) and the recurrence relation [16, p. 11] we get

$$\begin{aligned}&u\left((z-a)\,\psi_n(z)\,\overline{\phi_{n-1}}\left(\frac{1}{z}\right)\right)\\ &\quad = -\phi_{n+1}(0)\,e_n\,\overline{\phi_{n-1}(0)} - \frac{\phi'_n(a)}{\lambda + K^{(1,1)}_{n-1}(a,a)}\,u\left((z-a)\,K^{(0,1)}_{n-1}(z,a)\,\overline{\phi_{n-1}}\left(\frac{1}{z}\right)\right)\end{aligned}$$

$$= -\phi_{n+1}(0)\, e_n\, \overline{\phi_{n-1}(0)} + \frac{a\phi'_n(a)\,\overline{\phi'_{n-1}(a)}}{\lambda + K^{(1,1)}_{n-1}(a,a)}$$

$$- \frac{\phi'_n(a)}{\lambda + K^{(1,1)}_{n-1}(a,a)}\, u\left(z\,\overline{\phi_{n-1}}\left(\frac{1}{z}\right)\right.$$

$$\left.\times\left(\frac{\phi_{n-1}(z)\,\overline{\phi'_{n-1}(a)}}{e_{n-1}} + \frac{\phi_{n-2}(z)\,\overline{\phi'_{n-2}(a)}}{e_{n-2}}\right)\right)$$

with

$$u\left(z\,\overline{\phi_{n-1}}\left(\frac{1}{z}\right)\left(\frac{\phi_{n-1}(z)\,\overline{\phi'_{n-1}(a)}}{e_{n-1}} + \frac{\phi_{n-2}(z)\,\overline{\phi'_{n-2}(a)}}{e_{n-2}}\right)\right)$$

$$= -\phi_n(0)\,\overline{\phi_{n-1}(0)\,\phi'_{n-1}(a)} + \frac{e_{n-1}}{e_{n-2}}\,\overline{\phi'_{n-2}(a)}.$$

If we identify both expressions,

$$-\frac{\phi_{n+1}(a)\,\overline{\phi_{n-1}(a)}}{K_n(a,a)} - \frac{M_n\phi_n(a)\,\overline{\phi_{n-1}(a)}}{K_{n-1}(a,a)} + N_n e_{n-1}\,\overline{\phi_{n-2}(0)}$$

$$= -\phi_{n+1}(0)\, e_n\,\overline{\phi_{n-1}(0)} + \frac{\phi'_n(a)}{\lambda + K^{(1,1)}_{n-1}(a,a)}$$

$$\times\left(a\,\overline{\phi'_{n-1}(a)} + \phi_n(0)\,\overline{\phi_{n-1}(0)\,\phi'_{n-1}(a)} - \frac{e_{n-1}}{e_{n-2}}\,\overline{\phi'_{n-2}(a)}\right).$$

If $\phi_{n-2}(0) \neq 0$ we deduce N_n from the above expression. If $\phi_{n-2}(0) = 0$ then $\phi_{n-2}(z) = z\phi_{n-3}(z)$ and therefore we can multiply in (7) by $\overline{\phi_{n-2}}\,(1/z)$ and the method may be iterated.

Next, if we take into account the Szegő recurrence relation [5, 12]

$$\phi^{1)*}_{n-1}(z) = \frac{e^{1)}_{n-1}}{e^{1)}_{n-2}}\,\phi^{1)*}_{n-2}(z) + \overline{\phi^{1)}_{n-1}(0)}\,\phi^{1)}_{n-1}(z)$$

and substitute in the above expression (6), we obtain

THEOREM 2.8.

$$\psi_n(z) = S_{1,n}(z)\,\phi^{1)}_{n-1}(z) + S_{0,n}(z)\,\phi^{1)*}_{n-1}(z) \tag{9}$$

where

$$S_{1,n}(z) = z + M_n - N_n \frac{e_{n-2}^{1)}}{e_{n-1}^{1)}} \overline{\phi_{n-1}^{1)}(0)}$$

$$S_{0,n}(z) = \phi_n^{1)}(0) + N_n \frac{e_{n-2}^{1)}}{e_{n-1}^{1)}}.$$

Proof.

$$\begin{aligned}
\psi_n(z) &= \phi_n^{1)}(z) + M_n \phi_{n-1}^{1)}(z) + N_n \frac{e_{n-2}^{1)}}{e_{n-1}^{1)}} \\
&\quad \times \left(\phi_{n-1}^{1)*}(z) - \overline{\phi_{n-1}^{1)}(0)}\, \phi_{n-1}^{1)}(z) \right) \\
&= \left(z + M_n - N_n \frac{e_{n-2}^{1)}}{e_{n-1}^{1)}} \overline{\phi_{n-1}^{1)}(0)} \right) \phi_{n-1}^{1)}(z) \\
&\quad + \left(\phi_n^{1)}(0) + N_n \frac{e_{n-2}^{1)}}{e_{n-1}^{1)}} \right) \phi_{n-1}^{1)*}(z). \quad \blacksquare
\end{aligned}$$

Remark. This result improves the previous one obtained in this paper since there are only two essential parameters. Nevertheless the price is the use of the family $\{\phi_n^1(z)\}_{n \in \mathbb{N}}$ instead of $\{\phi_n(z)\}_{n \in \mathbb{N}}$. Moreover, a recurrence relation in a proper sense as is verified for MOPS on the unit circle (see [8]) can be given,

THEOREM 2.9. *The sequence $\{\psi_n(z)\}_{n \in \mathbb{N}}$ satisfies the three-term recurrence relation*

$$W(z, 2)\, \psi_{n+1}(z) + Y(z, 3)\, \psi_n(z) + Z(z, 4)\, \psi_{n-1}(z) = 0,$$

where the coefficients are polynomials of degrees $\partial W(z, 2) \leqslant 2$, $\partial Y(z, 3) \leqslant 3$, and $\partial Z(z, 4) \leqslant 4$.

Proof. Consider the system

$$\begin{aligned}
\psi_{n-1}(z) &= S_{1,n-1}(z)\, \phi_{n-2}^{1)}(z) + S_{0,n-1}(z)\, \phi_{n-2}^{1)*}(z) \\
\psi_n(z) &= (S_{1,n}(z) + S_{0,n}(z)\, \overline{\phi_{n-1}^{1)}(0)})\, z\phi_{n-2}^{1)}(z) \\
&\quad + (S_{1,n}(z)\, \phi_{n-1}^{1)}(0) + S_{0,n}(z))\, \phi_{n-2}^{1)*}(z) \\
\psi_{n+1}(z) &= (S_{1,n+1}(z) + S_{0,n+1}(z)\, \overline{\phi_n^{1)}(0)})\, z\phi_{n-1}^{1)}(z) \\
&\quad + (S_{1,n+1}(z)\, \phi_n^{1)}(0) + S_{0,n+1}(z))\, \phi_{n-1}^{1)*}(z) \\
&= ((S_{1,n+1}(z) + S_{0,n+1}(z)\, \overline{\phi_n^{1)}(0)})\, z^2 \\
&\quad + (S_{1,n+1}(z)\, \phi_n^{1)}(0) + S_{0,n+1}(z))\, \overline{\phi_{n-1}^{1)}(0)}\, z)\, \phi_{n-2}^{1)}(z) \\
&\quad + ((S_{1,n+1}(z) + S_{0,n+1}(z)\, \overline{\phi_n^{1)}(0)})\, \phi_{n-1}^{1)}(0)\, z \\
&\quad + (S_{1,n+1}(z)\, \phi_n^{1)}(0) + S_{0,n+1}(z)))\, \phi_{n-2}^{1)*}(z).
\end{aligned}$$

If we denote by

$$T_{1,n}(z) = S_{1,n}(z) + S_{0,n}(z)\,\overline{\phi_{n-1}^{1)}(0)}, \qquad \partial T_{1,n}(z) = 1,$$
$$V_{1,n}(z) = S_{1,n}(z)\,\phi_{n-1}^{1)}(0) + S_{0,n}(z), \qquad \partial V_{1,n}(z) \leqslant 1,$$

then

$$\begin{vmatrix} \psi_{n-1}(z) & S_{1,n-1}(z) & S_{0,n-1}(z) \\ \psi_n(z) & T_{1,n}(z)\,z & V_{1,n}(z) \\ \psi_{n+1}(z) & T_{1,n+1}(z)\,z^2 + V_{1,n+1}\,\overline{\phi_{n-1}^{1)}(0)}\,z & T_{1,n+1}(z)\,\phi_{n-1}^{1)}(0)\,z + V_{1,n+1}(z) \end{vmatrix} = 0$$

follows. Therefore,

$$\begin{aligned} &\psi_{n+1}(z)(S_{1,n-1}(z)\,V_{1,n}(z) - T_{1,n}(z)\,S_{0,n-1}(z)\,z) \\ &\quad - \psi_n(z)(S_{1,n-1}(z)\,T_{1,n+1}(z)\,\phi_{n-1}^{1)}(0)\,z + S_{1,n-1}(z)\,V_{1,n+1}(z) \\ &\quad - S_{0,n-1}(z)\,T_{1,n+1}(z)\,z^2 - S_{0,n-1}(z)\,V_{1,n+1}(z)\,\overline{\phi_{n-1}^{1)}(0)}\,z) \\ &\quad + \psi_{n-1}(z)(T_{1,n}(z)\,T_{1,n+1}(z)\,\phi_{n-1}^{1)}(0)\,z^2 + T_{1,n}(z)\,V_{1,n+1}(z)\,z \\ &\quad - V_{1,n}(z)\,T_{1,n+1}(z)\,z^2 - V_{1,n}(z)\,V_{1,n+1}(z)\,\overline{\phi_{n-1}^{1)}(0)}\,z) = 0, \end{aligned}$$

or equivalently $W(z, 2)\,\psi_{n+1}(z) + Y(z, 3)\,\psi_n(z) + Z(z, 4)\,\psi_{n-1}(z) = 0$. ∎

3. The Positive Definite case

If u is a positive definite functional, according to Theorem 2.2, the form $\langle\, ,\, \rangle$ is positive definite if and only if

$$\frac{\lambda + K_n^{(1,1)}(a, a)}{\lambda + K_{n-1}^{(1,1)}(a, a)} > 0$$

for all $n \geqslant 1$. The functional induced by the Lebesgue normalized measure, defined by $u(z^n) = \delta_{n0}$ $\forall n \in \mathbb{Z}$, belongs to this case. For such a measure according to (1), the following expression for the new orthogonal polynomials is obtained:

$$\begin{aligned} \psi_n(z) &= z^n - \frac{na^n \sum_{j=1}^{n-1} j(z\bar{a})^j}{\lambda + \sum_{j=1}^{n-1} j^2} = z^n - \frac{6na^n \sum_{j=1}^{n-1} j(z\bar{a})^j}{6\lambda + (n-1)\,n(2n-1)} \\ &= z^n - \frac{6n}{6\lambda + (n-1)\,n(2n-1)} \sum_{j=1}^{n-1} \frac{(n-j)\,z^{n-j}}{\bar{a}^j}. \end{aligned}$$

We remark that $\psi_n(0) = 0$ $\forall n \geqslant 1$ but there are no general results about the roots of $\psi_n(z)$. Indeed: if we consider $\psi_2(z) = z(z - 2/(\lambda + 1)\,\bar{a})$), its roots are $z_1 = 0$ and $z_2 = 2/((\lambda+1)\bar{a})$. Thus, if $\lambda > 1$ or $\lambda < -3$ then $|z_2| < 1$, if $\lambda = 1$ or $\lambda = -3$ then $|z_2| = 1$, and for $-3 < \lambda < 1$ it holds $|z_2| > 1$. Besides, if we denote by $E_n = \langle \psi_n(z), \psi_n(z) \rangle$, from the proof of Theorem 2.2

$$E_n = e_n + \frac{|\phi_n'(a)|^2}{\lambda + K_{n-1}^{(1,1)}(a,a)}, \qquad \text{and therefore} \quad E_n = \frac{2n^3 + 3n^2 + n + 6\lambda}{2n^3 - 3n^2 + n + 6\lambda}.$$

Returning now to the general situation, if we denote by $\{\varphi_n(z)\}_{n \in \mathbb{N}}$ and $\{\Psi_n(z)\}_{n \in \mathbb{N}}$ the sequences of orthonormal polynomials (OPS) related to u and $\langle\ ,\ \rangle$ respectively, then

THEOREM 3.1.

(i)
$$\Psi_n(z) = \left(\frac{e_n}{E_n}\right)^{1/2} \left(\varphi_n(z) - \frac{\varphi_n'(a)}{\lambda + K_{n-1}^{(1,1)}(a,a)} K_{n-1}^{(0,1)}(z,a)\right); \tag{10}$$

$$\text{where} \quad \varphi_n(z) = \left(\frac{1}{e_n}\right)^{1/2} z^n + \cdots$$

$$\text{and} \quad \Psi_n(z) = \left(\frac{1}{E_n}\right)^{1/2} z^n + \cdots;$$

(ii)
$$\Psi_n(z) = \left(\frac{E_n}{e_n}\right)^{1/2} \left(\varphi_n(z) - \frac{\varphi_n'(a)}{\lambda + K_n^{(1,1)}(a,a)} K_n^{(0,1)}(z,a)\right); \tag{11}$$

(iii)
$$\frac{e_n}{E_n} = 1 - \frac{|\varphi_n'(a)|^2}{\lambda + K_n^{(1,1)}(a,a)}; \tag{12}$$

(iv)
$$E_n - e_n = \frac{E_n\, |\varphi_n'(a)|^2}{\lambda + K_n^{(1,1)}(a,a)} = \frac{e_n\, |\varphi_n'(a)|^2}{\lambda + K_{n-1}^{(1,1)}(a,a)}; \tag{13}$$

(v)
$$|\Psi_n'(a)|^2 = \frac{\lambda^2 (1 - e_n/E_n)}{\lambda + K_{n-1}^{(1,1)}(a,a)}. \tag{14}$$

Proof. (i) By orthonormalizing in (1) we get (i), from which it follows

$$\Psi_n'(a) = \lambda \left(\frac{e_n}{E_n}\right)^{1/2} \frac{\varphi_n'(a)}{\lambda + K_{n-1}^{(1,1)}(a,a)}. \tag{15}$$

(ii) It is clear that $\Psi_n(z) = \sum_{k=0}^{n} a_k \varphi_k(z)$ with

$$a_j = u\left(\Psi_n(z)\, \varphi_j\left(\frac{1}{z}\right)\right) = -\lambda^{-1} \Psi_n'(a)\, \overline{\varphi_j'(a)} \qquad \text{for} \quad j = 0, \ldots, n-1,$$

and

$$a_n = u\left(\Psi_n(z)\, \overline{\varphi_n}\left(\frac{1}{z}\right)\right) = \left(\frac{E_n}{e_n}\right)^{1/2} - \lambda^{-1} \Psi_n'(a)\, \overline{\varphi_n'(a)},$$

from which

$$\Psi_n(z) = \left(\frac{E_n}{e_n}\right)^{1/2} \varphi_n(z) - \lambda^{-1} \Psi_n'(a)\, K_n^{(0,1)}(z, a).$$

If we take derivatives in this expression and evaluate at $z = a$ we get

$$\Psi_n'(a) = \lambda \left(\frac{E_n}{e_n}\right)^{1/2} \frac{\varphi_n'(a)}{\lambda + K_n^{(1,1)}(a, a)}. \tag{16}$$

Substituting (16) in the above expression (ii) holds.

(iii) From the proof of Theorem 2.2 we obtain

$$E_n = e_n \frac{\lambda + K_n^{(1,1)}(a, a)}{\lambda + K_{n-1}^{(1,1)}(a, a)},$$

and (iii) follows. Besides, it is straightforward that $0 < e_n/E_n \leqslant 1$ for $\lambda > 0$.

(iv) Also from the proof of Theorem 2.2 we get

$$e_n = E_n - \lambda^{-1} \psi_n'(a)\, \overline{\phi_n'(a)}.$$

Using orthonormal polynomials,

$$\left(\frac{e_n}{E_n}\right)^{1/2} = \left(\frac{E_n}{e_n}\right)^{1/2} - \lambda^{-1} \Psi_n'(a)\, \overline{\varphi_n'(a)}$$

holds. This implies that

$$\frac{e_n}{E_n} = 1 - \left(\frac{e_n}{E_n}\right)^{1/2} \lambda^{-1} \Psi_n'(a)\, \overline{\varphi_n'(a)}. \tag{17}$$

If we substitute this expression in (15) and (16) we deduce iv).

(v) Since $u(\Psi_n(z)\, \overline{\Psi_n}\,(1/z)) = \sum_{k=0}^{n} |a_k|^2$ then $\langle \Psi_n(z), \Psi_n(z) \rangle - \lambda^{-1} |\Psi_n'(a)|^2 = (e_n/E_n) + \lambda^{-2} |\Psi_n'(a)|^2 \sum_{j=0}^{n-1} |\varphi_j'(a)|^2$, that is, $1 - \lambda^{-1} |\Psi_n'(a)|^2 = (e_n/E_n) + \lambda^{-2} |\Psi_n'(a)|^2\, K_{n-1}^{(1,1)}(a, a)$, from which we conclude (v). ∎

4. Asymptotic Properties for Orthonormal Polynomials

Since u is a positive definite functional recall that there exists a finite positive Borel measure μ on the unit circle T such that

$$u\left(P(z)\, \bar{Q}\left(\frac{1}{z}\right)\right) = \int_T P(z)\, \overline{Q(z)}\, d\mu(z) \qquad \text{(see [8])}.$$

Let $dv(z) = |z-a|^2\, d\mu(z)$. Denote by $\{\varphi_n(z; v)\}_{n \in \mathbb{N}}$ the OPS related to v and by $\{K_n(z, y; v)\}_{n \in \mathbb{N}}$ the corresponding sequence of kernels. Recall that ([7])

$$\text{(i)} \qquad (z-a)\, \varphi_{n-1}(z; v) = \left(\frac{K_{n-1}(a, a)}{K_n(a, a)}\right)^{1/2} \times \left(\varphi_n(z) - \frac{\varphi_n(a)}{K_{n-1}(a, a)} \times K_{n-1}(z, a)\right), \tag{18}$$

where $(K_{n-1}(a, a)/K_n(a, a))^{1/2}$ can be written as $(1 + (|\varphi_n(a)|^2/K_{n-1}(a, a)))^{-1/2}$.

$$\text{(ii)} \qquad (z-a)\,\overline{(y-a)}\, K_{n-1}(z, y; v) = K_n(z, y) - \overline{K_n(y, a)}\,(K_n(a, a))^{-1}\, K_n(z, a). \tag{19}$$

THEOREM 4.1. *If the measure μ is such that $\{\phi_n(0)\} \to 0$ then*

$$\lim_{n \to \infty} \frac{|\varphi_n'(a)|^2}{K_{n-1}^{(1,1)}(a, a)} = 0. \tag{20}$$

Proof. From relations (18) and (19), if we take derivatives in (18) and evaluate at $z = a$ it follows that

$$\varphi_{n-1}(a; v) = \left(1 + \frac{|\varphi_n(a)|^2}{K_{n-1}(a, a)}\right)^{-1/2} \left(\varphi_n'(a) - \frac{\varphi_n(a)}{K_{n-1}(a, a)}\, K_{n-1}^{(1,0)}(a, a)\right). \tag{21}$$

In a similar way, if we take derivatives with respect to z in (19), evaluating in $z = a$ and taking complex conjugates we obtain

$$(y-a)\, K_{n-1}(y, a; v) = K_n^{(0,1)}(y, a) - \frac{K_n(y, a)}{K_n(a, a)}\, K_n^{(0,1)}(a, a).$$

Again, if we take derivatives with respect to y in the last expression and evaluate at "a" we get

$$K_{n-1}(a, a; v) = K_n^{(1,1)}(a, a) - \frac{|K_n^{(1,0)}(a, a)|^2}{K_n(a, a)},$$

and $K_{n-1}(a, a; v) \leqslant K_n^{(1,1)}(a, a)$ holds. If we denote by $c_n = (1 + |\varphi_n(a)|^2/K_{n-1}(a, a))^{-1}$, we deduce from (21) and the above inequality:

$$\frac{|\varphi_{n-1}(a; v)|^2}{K_{n-2}(a, a; v)} \geqslant \frac{c_n}{K_{n-1}^{(1,1)}(a, a)} \left|\varphi_n'(a) - \frac{\varphi_n(a)}{K_{n-1}(a, a)}\, K_{n-1}^{(1,0)}(a, a)\right|^2 \tag{22}$$

On the other hand,

$$\left|\varphi_n'(a)-\frac{\varphi_n(a)}{K_{n-1}(a,a)}K_{n-1}^{(1,0)}(a,a)\right|\frac{1}{(K_{n-1}^{(1,1)}(a,a))^{1/2}}$$
$$\geqslant\frac{|\varphi_n'(a)|}{(K_{n-1}^{(1,1)}(a,a))^{1/2}}-\frac{|\varphi_n(a)|\,|K_{n-1}^{(1,0)}(a,a)|}{K_{n-1}(a,a)(K_{n-1}^{(1,1)}(a,a))^{1/2}}. \tag{23}$$

Applying the Cauchy–Schwarz inequality, $|K_{n-1}^{(1,0)}(a,a)|^2\leqslant K_{n-1}^{(1,1)}(a,a)\times K_{n-1}(a,a)$ holds. Therefore,

$$0\leqslant\frac{|\varphi_n(a)|\,|K_{n-1}^{(1,0)}(a,a)|}{K_{n-1}(a,a)(K_{n-1}^{(1,1)}(a,a))^{1/2}}\leqslant\frac{|\varphi_n(a)|\,|K_{n-1}^{(1,0)}(a,a)|}{(K_{n-1}(a,a))^{1/2}\,|K_{n-1}^{(1,0)}(a,a)|}$$
$$=\frac{|\varphi_n(a)|}{(K_{n-1}(a,a))^{1/2}}.$$

Since $\{\phi_n(0)\}\to 0$ and $|a|=1$ recall that $\lim_{n\to\infty}|\varphi_n(a)|^2/K_{n-1}(a,a)=0$, and so we get, as an immediate consequence of the previous relation, that

$$\lim_{n\to\infty}\frac{|\varphi_n(a)|\,|K_{n-1}^{(0,1)}(a,a)|}{K_{n-1}(a,a)(K_{n-1}^{(1,1)}(a,a))^{1/2}}=0 \tag{24}$$

Similarly from (22),

$$\lim_{n\to\infty}\frac{1}{(K_{n-1}^{(1,1)}(a,a))^{1/2}}\times\left|\varphi_n'(a)-\frac{\varphi_n(a)}{K_{n-1}(a,a)}K_{n-1}^{(1,0)}(a,a)\right|=0, \tag{25}$$

taking into account that $\lim_{n\to\infty}c_n=1$ and $\lim_{n\to\infty}|\varphi_{n-1}(a,v)|^2/K_{n-2}(a,a;v)=0$. Therefore, the result in (20) follows immediately from (23) taking into account (24) and (25). ∎

COROLLARY 4.2. *If the measure μ is such that $\{\phi_n(0)\}\to 0$ and $\lambda>0$, then the following result holds*:

(i) $$\lim_{n\to\infty}\frac{e_n}{E_n}=1,$$

(ii) $$\lim_{n\to\infty}\Psi_n'(a)\,\overline{\varphi_n'(a)}=0,$$

(iii) $$\lim_{n\to\infty}\Psi_n'(a)=0,$$

(iv) $$\lim_{n\to\infty}(E_n-e_n)=0.$$

Proof. (i) Since

$$0 \leqslant \frac{|\varphi_n'(a)|^2}{\lambda + K_n^{(1,1)}(a, a)} \leqslant \frac{|\varphi_n'(a)|^2}{K_n^{(1,1)}(a, a)} \leqslant \frac{|\varphi_n'(a)|^2}{K_{n-1}^{(1,1)}(a, a)},$$

we deduce

$$\lim_{n \to \infty} \frac{|\varphi_n'(a)|^2}{\lambda + K_n^{(1,1)}(a, a)} = 0$$

and thus, (12) implies the result.

(ii) and (iii) Both results follow from (17) and (14), respectively, if we apply (i).

(iv) By using relation (13) if we take into account that

$$\lim_{n \to \infty} \frac{|\varphi_n'(a)|^2}{\lambda + K_{n-1}^{(1,1)}(a, a)} = 0$$

and $\lim_{n \to \infty} e_n \geqslant 0$, then the result follows. ∎

THEOREM 4.3. *Suppose μ is a measure such that $\{\phi_n(0)\} \to 0$ and $\lambda > 0$. Then*

$$\lim_{n \to \infty} \frac{\Psi_n(z)}{\varphi_n(z)} = 1 \qquad \text{uniformly for} \quad |z| > r > 1.$$

Proof. Relation (10) implies that

$$\frac{\Psi_n(z)}{\varphi_n(z)} = \left(\frac{e_n}{E_n}\right)^{1/2} \left(1 - \frac{\varphi_n'(a) K_{n-1}^{(0,1)}(z, a)}{(\lambda + K_{n-1}^{(1,1)}(a, a)) \varphi_n(z)}\right).$$

From the Cauchy–Schwarz inequality it follows that

$$|K_{n-1}^{(0,1)}(z, a)|^2 \leqslant K_{n-1}^{(1,1)}(a, a) K_{n-1}(z, z).$$

Therefore

$$\begin{aligned} 0 &\leqslant \left(\frac{e_n}{E_n}\right)^{1/2} \frac{|\varphi_n'(a) K_{n-1}^{(0,1)}(z, a)|}{|\lambda + K_{n-1}^{(1,1)}(a, a)| \, |\varphi_n(z)|} \\ &\leqslant \left(\frac{e_n}{E_n}\right)^{1/2} \frac{|\varphi_n'(a)| \, (K_{n-1}^{(1,1)}(a, a))^{1/2} \, (K_{n-1}(z, z))^{1/2}}{|\lambda + K_{n-1}^{(1,1)}(a, a)| \, |\varphi_n(z)|} \\ &\leqslant \left(\frac{e_n}{E_n}\right)^{1/2} \frac{|\varphi_n'(a)| \, (K_{n-1}(z, z))^{1/2}}{(K_{n-1}^{(1,1)}(a, a))^{1/2} \, |\varphi_n(z)|}. \end{aligned}$$

Since

$$\frac{K_{n-1}(z,z)}{|\varphi_n(z)|^2} = \frac{|\varphi_n^*(z)|^2 - |\varphi_n(z)|^2}{(1-|z|^2)\,|\varphi_n(z)|^2} = \frac{1}{(-1+|z|^2)}\left(1 - \frac{|\varphi_n^*(z)|^2}{|\varphi_n(z)|^2}\right)$$
$$< \frac{1}{(|z|^2-1)},$$

then $K_{n-1}(z,z)/|\varphi_n(z)|^2$ is bounded for $|z|>1$. On the other hand we have

$$\lim_{n\to\infty}\left(\frac{e_n}{E_n}\right)^{1/2} \frac{|\varphi_n'(a)|}{(K_{n-1}^{(1,1)}(a,a))^{1/2}} = 0.$$

By using these two last results we obtain that

$$\lim_{n\to\infty}\left(\frac{e_n}{E_n}\right)^{1/2} \frac{\varphi_n'(a)\,K_{n-1}^{(0,1)}(z,a)}{(\lambda + K_{n-1}^{(1,1)}(a,a))\,\varphi_n(z)} = 0 \qquad \text{for} \quad |z|>1$$

and the result follows. ∎

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